Mentors Eduserv®

IOQM 2024-25

Date : 08/09/2024 | **Duration: 3 Hours.** | **Maximum Marks: 100**

INSTRUCTIONS

1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.

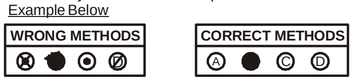
2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen**. Darken the bubbles completely, only after you are sure of your answer: else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The registration number and date of birth will be your login credentials for accessing your score.

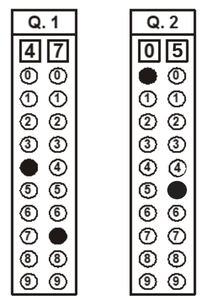
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.

INSTRUCTIONS
1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below



4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tempering the Original
7. Please do not make any stray marks on the answer sheet.



6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.

7. Questions **1 to 10 carry 2 marks** each; questions **11 to 20 carry 3 marks** each; questions **21 and 30 carry 5 marks** each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it further for verification purposes.

13. You may take away the question paper after the examination.

Name of Student :...

Batch : ....

Enrolment No. | | | | | | | | | | | | | | | | |

Note :

1. N denotes the set of all natural numbers, 1,2,3,.....

2. For a positive real number x, $\sqrt{x}$ denotes the positive square root of x. for example $\sqrt{4} = +2$.

3. Unless otherwise specified, all numbers are written in bas 10.

I O Q M

(INDIAN OLYMPIAD QUALIFIER IN MATHEMATICS)

PAPER WITH SOLUTION

[:Q.1] The smallest positive integer that does not divide $1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9$ is :

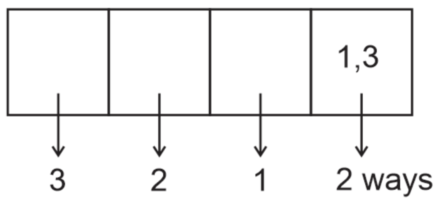
[:ANS] 11

[:SOLN] 11 is smallest prime number which are not divide the given number.

[:Q.2] The number of four-digit odd numbers having digits 1, 2, 3, 4, each occuring exactly once, is:

[:ANS] 12 days

[:SOLN] 1,2,3,4



$= 3\times 2\times 1\times 2 = 12$ ways

[:Q.3] The number obtained by taking the last two digits of 5^{2024} in the same order is:

[:ANS] 25

[:SOLN] 5^{2024}

$\underline{5^{n+1}}$ ends with 25, $n \in N$

$5^1 = 05$

$5^2 = \underline{25}$

$5^3 = 1\underline{25}$

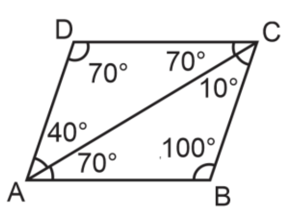
$5^4 = 6\underline{25}$

$5^5 = 31\underline{25}$

[:Q.4] Let ABCD be a quadrilateral with $\angle ADC = 70°$, $\angle ACD = 70°$, $\angle ACB = 10°$ and $\angle BAD = 110°$. The measure of $\angle CAB$ (in degrees) is:

[:ANS] 70

[:SOLN]



$\boxed{\angle CAB = 70°}$

$\angle DAC = 180° - (\angle ADC + \angle ACD)$

$= 40$

$\angle BAD = 110°$

then $\angle CAB = 110° - \angle DAC$

$= 110° – 40° = 70°$

[:Q.5] Let $a = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, let $b = \frac{x}{z} + \frac{y}{x} + \frac{z}{y}$ and let $c = \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right)$. The value of $|ab - c|$ is :

[:ANS] 1

[:SOLN] $a = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ $b = \frac{x}{z} + \frac{y}{x} + \frac{z}{y}$ and $c = \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right)$

Put x = y = z = 1

a = 1 + 1 + 1 = 3

b = 1 + 1 + 1 = 3

$c = 2 \times 2 \times 2 = 8$

$|ab - c| = |9 - 8| = 1$

[:Q.6] Find the number of triples of real numbers (a, b, c) such that $(a^{20} + b^{20} + c^{20} = a^{24} + b^{24} + c^{24} = 1$.

[:ANS] 6

[:SOLN] All possible triplets :

(1,0,0) (0,1,0), (0,0,1)

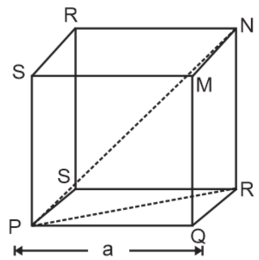
(–1,0,0) (0, –1, 0)

(0,0,–1)

[:Q.7] **De**termine the sum of all possible surface areas of a cube two of whose vertices are (1,2,0) and (3, 3, 2).

[:ANS] 99

[:SOLN] $\left.\begin{matrix} A(3,3,2) \\ B(1,2,0) \end{matrix}\right\} \Rightarrow d = AB = \sqrt{4+1+4} = 3$



$$\left.\begin{array}{l}\#PQ = a \\ \#PR = d_1 \\ \#PN = d_2\end{array}\right\} \Rightarrow \begin{array}{l} d_1^2 = 2a^2 \\ d_2^2 = d_1^2 + a^2 = 3a^2 \end{array}$$

A = Area = $6a^2$

$$\left.\begin{array}{l}\text{If } a = d = 3 \Rightarrow A = 6 \times 3^2 = 54 \\ \text{If } d_1 = d = 3 \Rightarrow A = \dfrac{6.d_1^2}{2} = 27 \\ \text{If } d_2 = d = 3 \Rightarrow A = \dfrac{6.d_2^2}{3} = 18\end{array}\right\} \text{ sum of Area} = 54 + 27 + 18 = 99$$

Ans. 99

[:Q.8] Let n be the smallest integer such that the sum of digits of n is divisible by 5 as well as the sum of digits of (n + 1) is divisible by 5. What are the first two digits of n in the same order?

[:ANS] 49

[:SOLN] Let S(n) denote sum of digits of n.

So S|(n) and

S| s(n+1).

If we do not have any carry is n + 1, then S(m + 1) = s(n) + 1 so condition cannot be satisfied.

So we must have unit digit 9. So

S(n + 1) = s(n) – 9 + 1 + 0

= s(n) – 8

Proceeding the same way least n can be 49999 so that

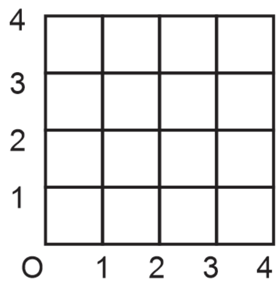
N + 1 = 50000.

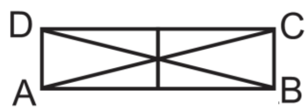
$\therefore$ Last two digits of n = 49.

[:Q.9] Consider the grid of points X = {(m, n) | 0 $\leq$ m, n $\leq$ 4}. We say a pair of points {(a,b), (c, d)} in X is a knight-move pair if (c = a $\pm$ 2 and d = b $\pm$ 1) or (c = a $\pm$ 1 and d = b $\pm$ 2). The number of knight-move pairs in X is:

[:ANS] 96

[:SOLN]





If C = a $\pm$ 2 & d = $\pm$ 1 $\Rightarrow$ Two consecutive smallest horizontal boxes moves from one corner to other

ie A $\rightleftharpoons$ C $\rightarrow$ 1 pair

OR

B $\rightleftharpoons$ D $\rightarrow$ 1 pair

So number of ways = $2 \times 3 \times 4 = 24$

Similarly for c = a $\pm$ 1 & b $\pm$ 2 = 24

Total = 48

[:Q.10] Determine the number of positive integral values of p for which there exists a triangle with sides a, b, and c which satisfy

$$a^2 + (p^2 + 9)b^2 + 9c^2 - 6ab - 6pbc = 0.$$

[:ANS] **5**

[:SOLN] $a^2 + (p^2 + 9)\, b^2 + 9c^2 - 6ab - 6pbc = 0$

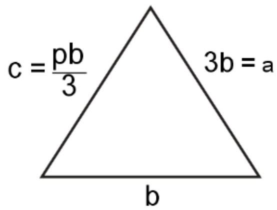
$(p^2b^2 - 6pbc + 9c^2) + (a^2 + 9b^2 - 6ab) = 0$

$(pb - 3c)^2 + (a - 3b)^2 = 0$

$\Rightarrow pb - 3c = 0 \; \& \; a - 3b = 0$

$$\boxed{c = \left(\frac{pb}{3}\right) \& \; a = 3b, p \in N.}$$

sides



$$\left.\begin{aligned} a+b>c \Rightarrow ub > \frac{pb}{3} \Rightarrow p<12 \\ b+c>a \Rightarrow b+\frac{pb}{3} > 3b \Rightarrow p>6 \end{aligned}\right\} \Rightarrow$$

$p \in (6,12)$ & $p \in N$.

p = {7, 8, 9, 10, 11}

[:Q.11] The positive real numbers a, b, c satisfy:

$$\frac{a}{2a+1}+\frac{2b}{3c+1}+\frac{3c}{a+1}=1$$

$$\frac{1}{a+1}+\frac{1}{2b+1}+\frac{1}{3c+1}=2$$

[:ANS] **12**

[:SOLN]

$$\left.\begin{aligned} \frac{a}{2b+1}+\frac{2b}{3c+1}+\frac{3c}{a+1}=1 \quad -(1) \\ \frac{1}{a+1}+\frac{1}{2b+1}+\frac{1}{3c+1}=2 \quad -(2) \end{aligned}\right\} \quad \text{Adding (1) and (2)}$$

$$\frac{a+1}{2b+1}+\frac{2b+1}{3c+1}+\frac{3c+1}{a+1}=3$$

Let $\frac{a+1}{2b+1}=\alpha, \frac{2b+1}{3c+1}=\beta, \frac{3c+1}{a+1}=\gamma$

$\alpha+\beta+\gamma=3$

$\alpha\cdot\beta\cdot\gamma=1$

A.m $\geq$ G.m

$$\frac{\alpha+\beta+\gamma}{3} \geq \sqrt[3]{\alpha\cdot\beta\cdot\gamma}$$

$1 \geq 1$

$$\therefore \frac{a+1}{2b+1}=\frac{2b+1}{3c+1}=\frac{3c+1}{a+1}=1$$

a +1 = 2b + 1

a = 2b –(i)

2b + 1 = 3c + 1

2b = 3c

a = 2b = 3c = K

$$\frac{1}{K+1}+\frac{1}{K+1}+\frac{1}{K+1}=2$$

$$\frac{3}{K+1}=2$$

3 = 2K + 2

$$\frac{1}{2}=K$$

$$a=\frac{1}{2}$$

$$b=\frac{1}{4}$$

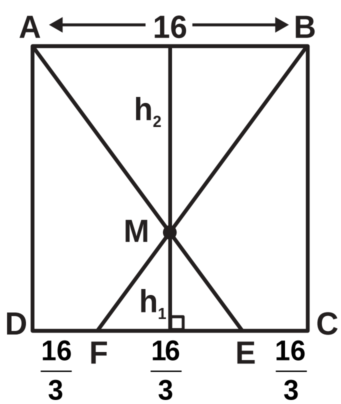
$$c=\frac{1}{6}$$

2 + 4 + 6 = 12

[:Q.12] Consider a square ABCD of side length 16. Let E, F be points on CD such that CE = EF = FD. Let the line BF and AE meet in M. The area of ΔMAB is:

[:ANS] 96 sq. units

[:SOLN]



$\Delta FME \sim \Delta BMA$

$$\frac{EF}{AB} = \frac{h_1}{h_1}$$

$$\frac{16/3}{16} = \frac{h_1}{h_2} \qquad \therefore \frac{h_1}{h_2} = \frac{1}{3}$$

as $h_1 + h_2 = 16$

x + 3x = 16

$\therefore x = 4$

$$\therefore ar(\Delta MAB) = \frac{1}{2} \times 16 \times 12 = 96 \text{ sq. units.}$$

[:Q.13] Three positive integers a, b, c with a > c satisfy the following equations:

ac + b + c = bc + a + 66, a + b + c = 32. Find the value of a.

[:ANS] **19**

[:SOLN] ac + b + c = bc + a + 66

ac – a = bc – c – b + 66

a(c – 1) = b(c – 1) –1(c – 1)+ 65

$$a = b - 1 + \frac{65}{c-1}$$

c – 1 can be 1 ×

5 ✓

13

65

when c – 1 = 1

a = b – 1 + 35

a = b + 64 ; but a + b + c = 32

×

when c – 1 = 5

a + b = 26

a = b + 12

a – b = 12

a + b = 16

a = 19

[:Q.14] Initially, there are 3^{80} particles at origin (0, 0). At each step the particles are moved to points above the x-axis as follows: if there are n particles at any (x, y), then $\left[\frac{n}{3}\right]$ of them are moved to (x + 1, y + 1), $\left[\frac{n}{3}\right]$ are moved (x, y + 1) and the remaining to (x – 1, y + 1). For example, after first step, there are 3^{79} particles each at (1, 1), (0, 1) and (–1, 1). After the second step, there are 3^{78} particles each at (–2, 2) and (2, 2), 2×3^{78} particles each at (–1, 2) and (1, 2), and 3^{79} particles at (0, 2). After 80 steps, the number of particles at (79, 80) is:

[:ANS] **80**

[:SOLN] Continuing the pattern after r^{th} step the number of points at (r, r) is 3^{80-r} while the number of points at (r – 1, r) is $r\times3^{80-r}$, therefore, after 80 steps the number of points at (79, 80) is 80.

[:Q.15] Let X be the set consisting of twenty positive integers n, n + 2, ..., n + 38. The smallest value of n for which any three numbers a, b, $c\in X$, not necessarily distinct, form the sides of an acute-angle triangle is:

[:ANS] **92**

[:SOLN] $a^2 + b^2 > c^2$

$\therefore n^2+n^2>(n+38)^2$

$n^2-76n>38^2$

$(n-38)^2>2\times38^2$

or $n>38+\sqrt{2}\times38$

$n>91.732$

$\therefore n=92$

[:Q.16] Let $f:R\to R$ be a function satisfying the relation 4f (3 – x) + 3f (x) = x^2 for any real x. Find the value of f(27) – f(25) to the nearest integer. (Here R denotes the set of real numbers.)

[:ANS] **8**

[:SOLN] Replace $x\to3-x$

$\therefore 12f(3-x)+9f(x)=3x^2$

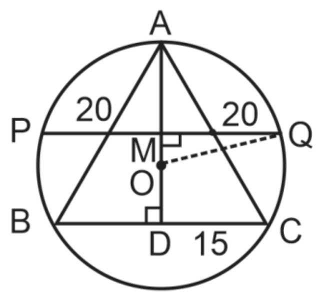
& $12f(3.x) + 16f(x) = 4(3 - x)^2$

$$\therefore f(x) = \frac{x^2 - 24x + 36}{7} \qquad \therefore f(27) - f(25) = 8.$$

[:Q.17] Consider an isosceles triangle ABC with sides BC = 30, CA = AB = 20. Let D be the foot of the perpendicular from A to BC, and let M be the midpoint of AD. Let PQ be a chord of the circumcircle of triangle ABC, such that M lies on PQ and PQ is parallel to BC. The length of PQ is:

[:ANS] **25**

[:SOLN]



Let O be the circumcenter then O lies on AD.

$$AD = \sqrt{20^2 - 15^2} = 5\sqrt{7}$$

$$AM = MD = \frac{5\sqrt{7}}{2}$$

$$\text{Circum radius, } R = \frac{abc}{4\Delta} = \frac{20 \times 20 \times 20}{4 \times \frac{1}{2} \times 30 \times 5\sqrt{7}} = \frac{40}{\sqrt{7}}$$

$$OM = OA - AM = \frac{40}{\sqrt{7}} - \frac{5\sqrt{7}}{2} = \frac{45}{2\sqrt{7}}$$

$$\therefore PA = 2MQ = 2\sqrt{OQ^2 - OM^2}$$

$$= 2\sqrt{\left(\frac{40}{\sqrt{7}}\right)^2 - \left(\frac{45}{2\sqrt{7}}\right)^2} = 25.$$

[:Q.18] Let p, q be two-digit numbers neither of which are divisible by 10. Let r be the four-digit number by putting the digits of p followed by the digits of q (in order). As p, q vary, a computer prints r on the screen if gcd(p, q) = 1 and p + q divides r. Suppose that the largest number that is printed by the computer is N. Determine the number formed by the last two digits of N (in the same order).

[:ANS] **13**

[:SOLN] r = 100 p + q

p + q | r

$\Rightarrow p+q \mid 100\,p+q$

$\Rightarrow p+q \mid (100\,p+q)-(p+q)$

$\Rightarrow p+q \mid 99p$

$(p,q)=1 \Rightarrow (p,p+q)=1$

p & p + q are coprime

$\therefore p+q \mid 99$

but p & q are two digit numbers

$\therefore p+q = 33 \text{ or } 99$

N = largest value of r

$= 100 \times 86 + 13$

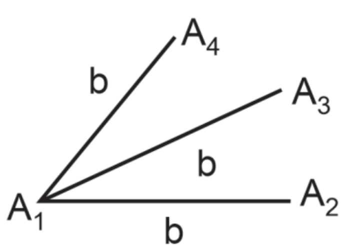
$= 8613$

last two digits of N = 13

[:Q.19] Consider five points in the plane, with no three of them collinear. Every pair of points among them is joined by a line. In how many ways can we color these lines by red or blue, so that no three of the points form a triangle with lines of the same color.

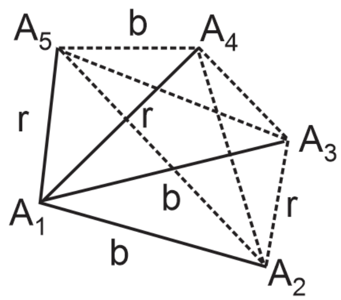
[:ANS] **12**

[:SOLN] Let A_1, A_2, A_3, A_4, A_5 be the 5 points. We cannot have 3 lines of same color emerging from same Ai.



For example if A_1A_2, A_1A_3 and A_1A_4 are blue then A_2A_3, A_3A_4, A_2A_4 must be all red but this will make sides of $A_2A_3A_4$ to be all red.

So from every A_i, two blue & two red lines should emerge.



In particular, if A_1A_2, A_1A_3 are colored blue and remaining two red, then A_2A_3 must be red and A_4A_5 must be blue. A_3A_4 can be blue or red. After that only one way to color remaining lines.

So total number of colorings $= {}^4C_2 \times 2 = 12$

[:Q.20] On a natural number n you are allowed two operations: (1) multiply n by 2 or (2) subtract 3 from n. For example starting with 8 you can reach 13 as follows: $8 \to 16 \to 13$. You need two steps and you cannot do in less than two steps. Starting from 11, what is the least number of steps required to reach 121?

[:ANS] **10**

[:SOLN] Instead of going from 11 to 121, think of going from 121 to 11 & it can be achieved as following

$$121 \xrightarrow{+3} 124 \xrightarrow{\times\frac{1}{2}} 62 \xrightarrow{\times\frac{1}{2}} 31 \xrightarrow{+3} 34 \xrightarrow{\times\frac{1}{2}} 17 \xrightarrow{+3} 20 \xrightarrow{\times\frac{1}{2}} 10 \xrightarrow{\times\frac{1}{2}}$$
$$5 \xrightarrow{+3} 8 \xrightarrow{+3} 11$$

a total of 10 steps.

[:Q.21] An integer n is such that $\left\lfloor \frac{n}{9} \right\rfloor$ is a three digit number with equal digits, and $\left\lfloor \frac{n-172}{4} \right\rfloor$ is a 4 digit number with the digits 2,0,2,4 in some order. What is the remainder when n is divided by 100 ?

[:ANS] 91

[:SOLN] $111 \le \left\lfloor \frac{n}{9} \right\rfloor \le 999$ and $2024 \le \left\lfloor \frac{n-172}{4} \right\rfloor \le 4220$

$\Rightarrow 111 \le \frac{n}{9} < 1000$ and $2024 \le \frac{n-172}{4} < 4221$

$\Rightarrow 999 \le n < 9000$ and $8268 \le 4221 \times 4 + 172$

$\Rightarrow 8268 \le n < 9000$

$\Rightarrow 918\frac{6}{9} \le \frac{n}{9} < 1000$

$\Rightarrow \left\lfloor \frac{n}{9} \right\rfloor = 999$

$\Rightarrow 999 \le \frac{n}{9} < 1000$

$8991 \le n < 9000$ - (1)

$\Rightarrow 2204\frac{3}{4} \le \frac{n-172}{4} < 2207$

$\therefore \left\lfloor \frac{n-172}{4} \right\rfloor = 2204$

$\Rightarrow 8988 \le n < 8992$ - (2)

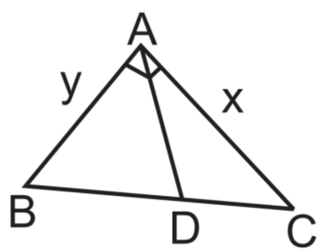
from (1) & (2) n = 8991

$\therefore$ remainder =100

[:Q.22] In a triangle ABC, $ABC, \angle BAC = 90°$. Let D be the point on BC such that AB + BD = AC + CD. Suppose BD : DC = 2 : 1. If $\frac{AC}{AB} = \frac{m+\sqrt{p}}{n}$, where m, n are relatively prime positive integers and p is a prime number, determine the value of m + n + p.

[:ANS] 34

[:SOLN]



Let $AC = x.2\,AB = y$

$\Rightarrow BC = \sqrt{x^2 + y^2}$

as $\frac{BD}{DC} = \frac{2}{1}$

$BD = \frac{2}{3}\sqrt{x^2 + y^2}$

$2DC = \frac{1}{3}\sqrt{x^2 + y^2}$

Now AB + BD =AC + CD

$y + \frac{2}{3}\sqrt{x^2 + y^2} = x + \frac{1}{3}\sqrt{x^2 + y^2}$

$\frac{1}{3}\sqrt{x^2 + y^2} = x - y$

$\frac{1}{3}\sqrt{\left(\frac{x}{y}\right)^2 + 1} = \frac{x}{y} - 1$

Let $\frac{x}{y} = t$, clearly t > 1

$\frac{1}{9}\left(t^2 + 1\right) = \left(t - 1\right)^2$

$\Rightarrow 8t^2 - 18t + 8 = 0$

$4t^2 - 9t + 4 = 0$

$t = \frac{9 \pm \sqrt{17}}{8}$

$\therefore \frac{AC}{AB} = \frac{9 + \sqrt{17}}{8}$

$\therefore$m + n+ p = 34

[:Q.23] Consider the fourteen number, $1^4, 2^4, ..., 14^4$. The smallest natural number n such that they leave distinct remainders when divided by n is :

[:ANS] 31

[:SOLN] Let r > s and r, s $\in$ {1, 2, … 14} such that $r^4 = s^4$ mod n

$\Rightarrow n | r^4 - s^4$

$\Rightarrow n | (r - s)(r + s)(r^2 + s^2)$

clearly $r - s \in \{1, 2, ..., 13\}$

$r + s \in \{3, 4, ..., 27\}$

So desired n $\geq$ 28.

For n = 28, we can take r = 13, s = 1

so that (r + s) (r – 1) = 14×12 which is divisible by n.

For n = 29, we can take r = 10 & s = 4 so that $r^2 + s^2 = 116$ which is divisible by 29.

Similarly n cannot be 30.

Now we check n = 31.

The remainders when $1^2, 2^2, \ldots, 14^2$ are divided by 31 are respectively,

1, 4, 9, 16, 25, 5, 18, 2, 19, 7, 28, 20, 20, 14, 10

We can seet that $r^2 + s^2$ is never divisible by 31.

So least value of n = 31.

[:Q.24] Consider the set F of all polynomials whose coefficients are in the set of {0,1}. Let $q(x) = x^3 + x + 1$.The number of polynomials p(x) in F of degree 14 such that the product p (x) q(x) is also in F is :

[:ANS] 50

[:SOLN] $p(x) \cdot q(x) = \left(x^{14} + ...\right)\left(x^3 + x + 1\right)$

$p(x) = x^{14} \rightarrow 1\text{case}$

$p(x) = x^{14} + x^{\alpha}$

$\alpha = 10, 9, 8, ...0 \rightarrow 11\text{Case}$

$p(x) = x^{14} + x^{\alpha} + x^{\beta}$

$$\left.\begin{array}{l} \alpha = 10, \beta = 6,5,4,3,2,1,0 \\ \alpha = 9, \beta = 5,4,3,2,1,0 \\ \alpha = 8, \beta = 4,3,2,1,0 \\ \alpha = 7, \beta = 3,2,1,0 \\ \alpha = 6, \beta = 2,1,0 \\ \alpha = 5, \beta = 1,0 \\ \alpha = 4, \beta = 0 \end{array}\right\} 28\,\text{case}$$

$p(x) = x^{14} + x^{\alpha} + x^{\beta} + x^{\alpha}$

$$\left.\begin{array}{l} \alpha = 10, \beta = 6, r = 2,1,0 \\ \alpha = 10, \beta = 5, r = 1,0 \\ \alpha = 10, \beta = 4, r = 0 \end{array}\right\} 6\,\text{Cases}$$

$\left.\begin{array}{l}\alpha=9,\beta=5,r=1,0\\ \beta=4,r=0\end{array}\right\}3\text{Case.}$

$\alpha=8,\beta=4,r=0\}1\text{Case}$

1 + 11 + 28 + 6 + 3 +1 = 50 Case

[:Q.25] A finite set M of positive integers consists of distinct perfect squares and the number 92. The average of the number in M is 85. If we remove 92 from M, the average drops to 84. If N^2 is the largest possible square in M, what is the value of N ?

[:ANS] 24

[:SOLN] Let $M=\{n_1^2,n_2^2,....,n_m^2,92\}$

$\therefore \frac{n_1^2+n_2^2+.....+n_m^2+92}{m+1}=85$

$\Rightarrow n_1^2+n_2^2+....+n_m^2=85m-7$ - (1)

Also $\frac{n_1^2+n_2^2+....+n_m^2}{m}84$

$\Rightarrow n_1^2+n_2^2+.....+n_m^2=84m$ - (2)

From (1) & (2)

85 m – 7 = 84 m $\Rightarrow$m =7

$\therefore n_1^2+n_2^2+....+n_7^2=84\times7=588$

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 7^2 + 22^2$

$\therefore$ largest value of n_7 = 22.

[:Q.26] The sum of [x] for all real numbers x satisfying the equation $16+15x+15x^2=\lfloor x\rfloor^3$ is :

[:ANS] 16

[:SOLN] $\because$ $16+15x+15x^4$ has D $15^2-4\times15<0$

$\therefore$ $16+15x+15x^2>0\forall x$

$\therefore \lfloor x\rfloor^3>0\Rightarrow x>0$

If $\lfloor x\rfloor=n$, then

$16+15n+15n^2\leq n^3$

$\Rightarrow 15\left(n^2+n+1\right)\le n^3-1$

$\Rightarrow 15\le n-1\Rightarrow n\ge 16$

Also $16+15(n+1)+15(n+1)^2>n^3$

$\Rightarrow n^3-15n^2<45n+46$

$\Rightarrow n^3-15n^3\le 45(n+1)$

$\Rightarrow \dfrac{n^2(n-15)}{n+1}\le 45$

$\Rightarrow n^2-16n+16-\dfrac{16}{n+1}\le 45$

$\Rightarrow n^2-16n+16<46$

$1\Rightarrow n(n-16)<30$

$\Rightarrow n\le 17$

For n = 16,

$16+15x+15x^2=16^3$

$\Rightarrow 15x(n+1)=16^3-16=16\times 15\times 17$

$\Rightarrow x=16$

For n = 17

$16+15x+15x^2=17^3$,

$P(x) = 15x^2 + 15x + 16 - 17^3$

Then p(17) P(18) < 0

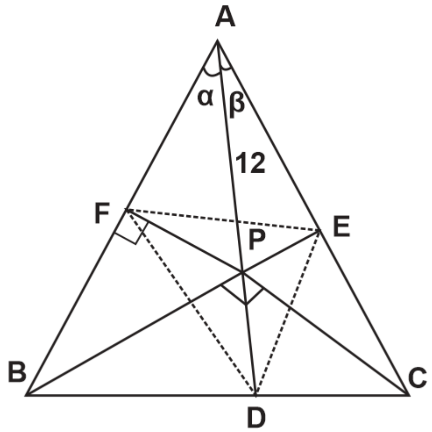
$\therefore$ a solutions in (17, 18)

$\therefore$ Sum of [x] = 16 + 17 = 33.

[:Q.27] In a triangle ABC, a point P in the interior of ABC is such that $\angle BPC-\angle BAC=\angle CPA-\angle CBA=\angle APB-\angle ACB$. Suppose $\angle BAC=30^\circ$ and AP = 12. Let D, E, F be the feet of perpendiculars form P on to BC, CA, AB respectively. If $m\sqrt{n}$ is the area of the triangle DFE where m, n are integers with n prime, then what is the value of the product mn?

[:ANS] 27

[:SOLN]



$\angle BPC - \angle BAC = \angle CPA - \angle CBA = \angle APB - \angle ACB = \theta$ ('say')

$\angle BPC + \angle CPA + \angle APB = 3\theta + \angle BAC + \angle CBA + \angle ACB.$

$360^o = 3\theta + 180^o \Rightarrow \theta = 60^o$

further $\angle BAC = 30^o \Rightarrow \angle BPC = 90^o$

let $\angle PAF = \alpha$ & $\angle PAE = \beta$

AFPE is cyclic quadrilateral

$12\sin\alpha.12\cos\beta + 12\sin\beta.12\cos\alpha = 12 \times EF$

$\Rightarrow EF = 12\sin(\alpha+\beta) = 12\sin 30^o = 6.$

let $\angle PBD = x \Rightarrow \angle PCD = 90 - x$

$\therefore \angle PBF + \angle PDE = B - x + c - (90 - x)$

$\angle FDE = B + C - 90^o = 60^o$

$[DEF] = \frac{\sqrt{3}}{4}EF^2 = 9\sqrt{3}$

$\therefore mn = 27.$

[:Q.28] Find the largest positive integer n < 30 such that $\frac{1}{2}\left(n^8 + 3n^4 - 4\right)$ is not divisible by the square of any prime number.

[:ANS] 20

[:SOLN] $\frac{1}{2}\left(n^8 + 3n^4 - 4\right) = \frac{1}{2}\left(n^4 + 4\right)\left(n^4 - 1\right)$

$$=\frac{1}{2}\left((n-1)(n+1)\left(n^2+1\right)\left(n^2-2n+2\right)\left(n^2+2n+2\right)\right)$$

$$=\frac{1}{2}\left((n-1)(n+1)\left(n^2+1\right)\left((n-1)^2+1\right)(n+1)^2+1\right)$$

Now, we w and largest positive integer n, n < 30 such that above expression is not divisible by square of any prime number. Clearly n should not be odd as n – 1, n + 1 & n^2+1 will be even & thus the expression will be divisible by 2^2

Now for n = 28; n – 1 = 27, therefore, expression is divisible by 3^2

for n = 26; n + 1 = 27, therefore, expression is divisible by 3^2

for n = 24; n + 1 = 25, therefore, expression is divisible by 5^2

for n = 22; $\left(n^2+1\right)\left((n+1)^2+1\right)=485\times530$, therefore, expression is divisible by 5^2

for n = 20; The expression is $\frac{1}{2}\times19\times21\times401\times362\times442$ which is not divisible by square of any prime

[:Q.29] Let $n=2^{19}3^{12}$. Let M denote the number of positive divisors of n^2 which are less then n but would not divide n. What is the number formed by taking the last two digits of M (in the same order)?

[:ANS] 228

[:SOLN] Let $2^p.3^q$ be divisor of n^2 which is less than n but does not divide n.

Case I … $P\in\{20,21,......,38\},q\in\{0,1,...,11\}$

$2^p.3^q<2^{19}.3^{12}$

$\Rightarrow 2^{p-19}<3^{13-q}$

Let p – 19 = a, 12 – q = b

$\therefore a\in\{1,2,...,19\},b\in\{1,2,..12\}$

$2^a<3^b$ –(1)

Case II $P\in\{0,1,2,...,18\},\ q\in\{13,14,...,24\}$

$2^p.3^q<2^{19}.3^{12}$

$\Rightarrow 3^{q-12}<2^{19-p}$

let q – 12 = c, 19 – p = d

$\therefore c \in \{1,2,...,12\},\ d \in \{1,2,...,19\}$

3c < 2d –(2)

If we compare the two cases for each (x, y) such that $x \in \{1,2,...,12\}$ & $y \in \{1,2,...,19\}$

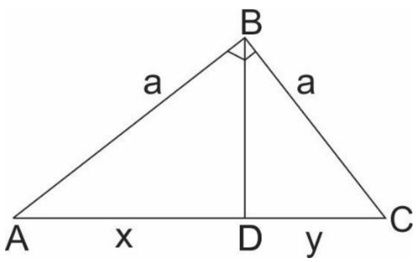
exactly one of (1) and (2) is satisfied.

So no. of required pairs = 12 x 19 = 228.

[:Q.30] Let ABC be a right – angled triangle with $\angle B = 90°$. Let the length of the altitude BD be equal to 12. What is the minimum possible length of AC, given that AC and the perimeter of triangle ABC are integers?

[:ANS] 25

[:SOLN]



ΔABC is a right angled triangle with

$\angle B = 90°, BD = 12$

Given that AC & AB + BC + AC are

Integers ie x+ y & a + c+ x +y are integers

$\Rightarrow a + c$ has to be an integer

Now $a = \sqrt{y^2 + 144}$ & $c = \sqrt{x^2 + 144}$

$\therefore a + c = \sqrt{y^2 + 144} + \sqrt{x^2 + 144}$ ______________ (1)

Further $xy = BD^2 = 144$

$\therefore (x, y) \equiv (16, 9)$ or (9,16)