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RMO - 2018

QUESTIONS & SOLUTIONS

October 07, 2018

Time : 3 hours

Instructions :

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. **Maximum marks : 102.**
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let ABC be a triangle with integer sides in which AB < AC. Let the tangent to the circumcircle of triangle ABC at A intersect the line BC at D. Suppose AD is also an integer. Prove that gcd (AB, AC) > 1.

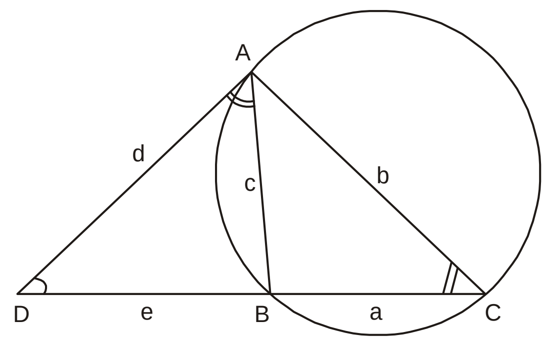
1. Let BC = a, CA = b, AB = c,

AD = d and BD = e,

Then a,b,c,d are intgers and c < b.

We are to prove that gcd (c,b) > 1.

If possible. let us suppose that gcd(c,b) = 1.



In ΔABD and ΔCAD,

$\angle DAB = \angle CAD$ (By alternate segment theorem)

$\angle D = \angle D$ (Common)

$\therefore$ by A A similarity criteria,

$\Delta ABD \sim \Delta CAD$

$$\therefore \frac{AB}{CA} = \frac{BD}{AD} = \frac{AD}{CD}$$

$$\Rightarrow \frac{c}{b} = \frac{e}{d} = \frac{d}{a+e} \qquad ....(i)$$

Now $\frac{c}{b} = \frac{e}{d} \Rightarrow e = \frac{cd}{b}$ ....(ii)

Again, $\frac{c}{b} = \frac{d}{a+e} \Rightarrow c(a+e) = bd$

$\Rightarrow c(a+\frac{cd}{b}) = bd$ (using (ii))

$\Rightarrow c(ab+cd) = b^2 d$

$\Rightarrow abc = d(b^2 - c^2) = d(b-c)(b+c)$

But gcd (c,b) = 1

$\therefore b+c \,|\, a$

$\Rightarrow b+c < a$

Which is a contradiction as a,b,c are side lengths of a triangle.

$\therefore$ we must have gcd (c,b) = 1.

2. Let n be a natural number. Find all real numbers x satisfying the equation $\sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}$.

Sol. Clearly x = 0 does not satisfy the given equation.

Let $x \neq 0$, then we have $\frac{kx^k}{1+x^{2k}} = \frac{k}{x^k + \frac{1}{x^k}}$

But $x^k + \frac{1}{x^k} \in (-\infty, -2] \cup [2, \infty) \ \forall x \in R$

$\therefore \quad \frac{k}{\frac{1}{x^k} + x^k} \leq \frac{k}{2} \ \forall x \in R.$

$\therefore \quad \sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} \leq \sum_{k=1}^{n} \frac{k}{2} = \frac{1}{2} \cdot \frac{n(n+1)}{2}$

Where equality holds iff $\frac{kx^k}{1+x^{2k}} = \frac{k}{2} \ \forall k \in \{1, ..., n\}$

i.e. $1+x^{2k}=2x^k$

i.e. $\left(x^k-1\right)^2=0\ \forall k\in\{1,....,n\}$

i.e. x = 1

So the given equation is satisfied only for x = 1.

3. For a rational number r, its period is the length of the smallest repeating block in its decimal expansion. For example, the number r = 0.123123123... has period 3. If S denotes the set of all rational numbers r of the form r = 0. $\overline{abcdefgh}$ having period 8, find the sum of all the elements of S.

Sol. First we show that if $r=0.\overline{abcdefgh}$ has period 8, then 1 – r also has period 8.

Let $a'=9-a$, $b'=9-b$, etc.

$\because\ 1=0.999...$

$\therefore\ 1-r=0.\overline{a'b'c'd'e'f'g'h'}$ which is a rational number of period 8.

Now number of numbers of the form $0.\overline{a\,b\,c\,d\,e\,f\,g\,h}$ is 10^8 since each digit a, b, ...,h can be selected in 10 ways.

But this includes the numbers of the form $0.\overline{a\,b\,c\,d\,a\,b\,c\,d}$ whose number is 10^4.

$\therefore$ number of numbers of the from $0.\overline{a\,b\,c\,d\,e\,f\,g\,h}$ with period 8 = 10^8-10^4.

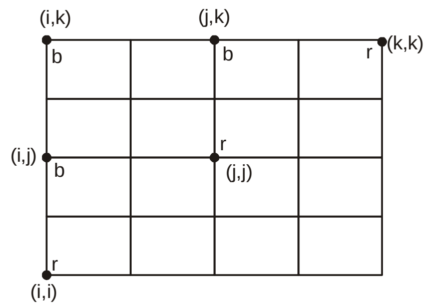
Now dividing these numbers into pairs of the form (r, 1–r), we get the sum of all these numbers

$$=\frac{10^8-10^4}{2}=5\times10^3\left(10^4-1\right).$$

4. Let E denote the set of 25 points (m,n) in the xy-plane, where m,n are natural numbers, $1\le m\le 5, 1\le n\le 5.$ Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form (a,b), (a + k,b), (a + k, b + k), (a, b + k) for some positive integer k such that at least three of these four points have the same colour. (That is, there alway exist four points in the set E which form the vertices of a square with sides parallel to axes and having at least three points of the same colour.)

Sol. Suppose there does not exists any square with vertices in E and sides parallel to axes such that 3 or more of its vertices have the same colour. Consider the five points (1,1), (2,2), (3,3), (4,4), (5,5). Among the five points, 3 or more must have same colour by pegion hole principle. Without loss of generality, let us suppose that (i,i), (j,j) and (k,k) are coloured red, where $1\le i<j<k\le 5$. Since (i,i), (i,j) and (j,j) are three vertices of a square with sides parallel to axes, (i,j) must be coloured blue. Similarly (i,k) and (j,k) must also be coloured blue.

Case 1 : If i + k = 2j, then (i,j), (i,k), (j,k), (j,j) from vertices of a square of which 3 are colored blue.



Case 2 : If $i+k \neq 2j$, then $1 \leq i+k-j \leq i+5-j \leq 5$ also $i+k-j \neq j$.

In this case we have

(i,j), (i,k), (i + k - j,k), (i + k – j, j) to be four vertices of a square

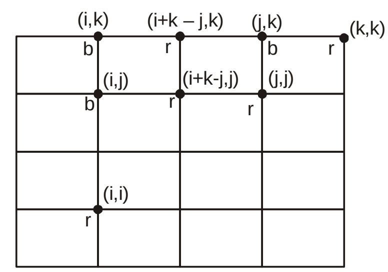
so (i + k – j,k) and (i + k – j, j) must be coloured red.

But then we get (j,j), (j,k), (i + k – j,k) and (i + k – j, j) to

be four vertices of a square of which 3 are red.

Thus in both cases we arive at a contradiction.

Hence the result.



5. Find all natural numbers n such that $1+[\sqrt{2n}]$ divides 2n. (For any real number x, [x] denotes the largest integer not exceeding x.)

Sol. Let $[\sqrt{2n}] = k, k \in I^+$,

then $k \leq \sqrt{2n} < k+1$

$\Rightarrow k^2 \leq 2n < (k+1)^2$

$\therefore\ 2n = (k+1)^2 - r$, where $r \in \{1, 2, ..., 2k+1\}$

Now $1+[\sqrt{2n}]\ |\ 2n$

$\Rightarrow 1+k \,|\, (k+1)^2 - r$

$\Rightarrow 1+k \,|\, r$

But $1 \leq r \leq 2k+1$

$\therefore r = k+1$

$\therefore 2n = (k+1)^2 - (k+1) = k(k+1)$

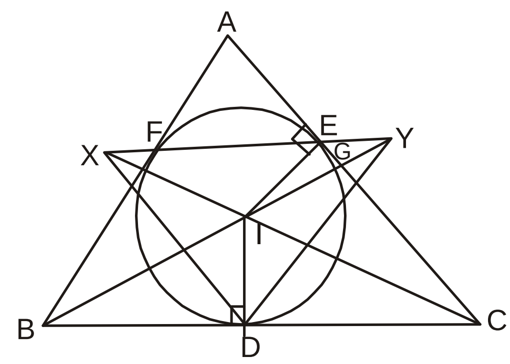
$\therefore n = \dfrac{k(k+1)}{2}$, where $k \in I^+$

6. Let ABC be an acute-angled triangle with AB < AC. Let I be the incentre of triangle ABC, and let D,E,F be the points at, which its incircle touches the sides BC, CA, AB respectively. Let BI, CI meet the line EF at Y,X respectively. Further assume that both X and Y are outside the triangle ABC. Prove that

(i) B,C,Y,X are concyclic, and (ii) I is also the incentre of triangle DYX.

Sol. Let BY meet AC at G.

$\because$ AE and AF are tangents to the incircle



$\therefore \quad \angle AEF = \angle AFE = 90^o - \frac{\angle A}{2}$

$\therefore \quad \angle YEG = \angle AEF = 90^o - \frac{\angle A}{2}$

Also $\angle EGY = \angle BGC = 180^o - \left(\frac{\angle ABC}{2} + \angle ACB\right)$

$\therefore \quad \angle EYG = 180^o - (\angle YEG + \angle EGY)$

$= 180^o - \left(90^o - \frac{\angle A}{2} + 180^o - \left(\frac{\angle ABC}{2} + \angle ACB\right)\right)$

$= \frac{\angle A}{2} + \frac{\angle ABC}{2} + \angle ACB - 90^o$

$= \frac{\angle ACB}{2} = \angle XCB$

$\therefore \quad \angle BYX = \angle EYG = \angle XCB$

$\therefore$ B, C, Y, X are concyclic. ... (i)

Also $\angle EYI = \angle EYG = \frac{\angle ACB}{2} = \angle ECI$

$\therefore$ Points E, I, C, Y are also concylcic.

But $\angle IDC = \angle IEC = 90^o$

$\therefore$ D also lie on the circle through E, I, C, Y.

$\therefore \quad \angle IYD = \angle ICD = \frac{\angle ACB}{2} = \angle EYG = \angle XYI$

$\therefore$ YI is bisector of $\angle XYD$.

Similarly, XI is also bisector of $\angle YXD$.

Hence I is incentre of ΔDXY ... (ii)