PRMO - 2018

QUESTIONS & SOLUTIONS

19-08-2018

1. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number of pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n?

1. **(17)**

Let 1st volume contains 'n_1' number of pages

2nd volume starts from $n_1 + 1$ and end at $2n_1 + 50$

3rd volume starts from $2n_1 + 51$

$\therefore\ 1 + n_1 + 1 + 2n_1 + 51 = 1709$

$3n_1 + 53 = 1709$

$n_1 = 552$

1st volume 1 to 552

2nd volume 553 to 1154

3nd volume 1155 to 2057

$2057 = 11 \times 187 = 11 \times 11 \times 17$

2. In a quadrilateral ABCD, it is the given that AB = AD = 13, BC = CD = 20, BD = 24. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r?

2. **(8)**

BE = DE = 12

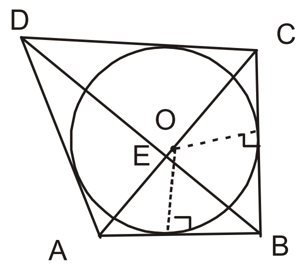
AB = AD = 13

BC = CD = 20

$\therefore AE = 5\ \&\ CE = 16$

Let O be the centre of the circle

$\therefore ar(\Delta ABC) = ar(\Delta OAB) + ar(\Delta OBC)$



$$\frac{1}{2} \times 21 \times 12 = \frac{1}{2} 13 \cdot r + \frac{1}{2} 20 \cdot r$$

$$r = \frac{21 \times 12}{33} = \frac{84}{11}$$

$\therefore$ Integer closest to r is 8.

3. Consider all 6-digit numbers of the form abccba where b is odd. Determine the number of all such 6 -digit numbers that are divisible by 7.

3. **(70)**

For abccba to be divisible by 7, cba – abc must be divisible by 7.

$\therefore$ 99(c – a) must be divisible by 7 or c – a must be divisible by 7.

Now b $\in$ {1,3,5,7,9} and (c,a) $\in$ {(1,1), (2,2),(3,3), (4,4), (5,5), (6,6), (7,7),(8,8), (9,9) (9,2),(2,9),(8,1),(1,8), (0,7)}

$\therefore$ Total number of required numbers is 70.

4. The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is, 1,6,5,8,9,0 are digits in base b in the above equation). Find the sum of all possble values of $b \geq 10$ satisfying the equation.

4. **(12)**

$166 \times 56 = 8590$

$\Rightarrow (6 + 6b + b^2)(6 + 5b) = (9b + 5b^2 + 8b^3)$

$\Rightarrow 3b^3 - 31b^2 - 57b - 36 = 0$

$\Rightarrow b = 12$

5. Let ABCD be a trapezium in which AB || CD and $AD \perp AB$. Suppose ABCD has an incircle which touches AB at Q and CD at P. Given that PC = 36 and QB = 49. find PQ.

5. **(84)**

We have BN = QB = 49

CN = CP = 36

$\therefore BC = BN + CN = 85$

Now,

$CB^2 = (CT)^2 + (BT)^2$

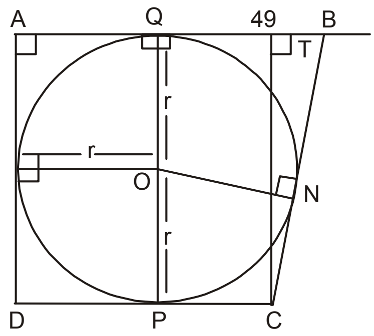
$(85)^2 = (PQ)^2 + (13)^2$

$\therefore PQ^2 = 85^2 - 13^2$

$PQ^2 = 72 \times 98$

PQ = 2 × 36 × 2 × 49

PQ = 2 × 6 × 7 = 84



6. Integers a,b,c satisfy a + b – c = 1 and $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$?

6. **(18)**

$a, b, c \in I$

$a + b - c = 1$ & $a^2 + b^2 - c^2 = -1$

$\Rightarrow a + b = ab \quad \Rightarrow a + b - ab = 0$

$\Rightarrow (1-a)(1-b) = 1 \qquad \Rightarrow 1 - a = 1 \,\&\, 1 - b = 1 \Rightarrow a = 0 \,\&\, b = 0 \Rightarrow c = -1$

or $1 - a = -1 \,\&\, 1 - b = -1 \Rightarrow a = 2 \,\&\, b = 2 \Rightarrow c = 3$

So, (a ,b, c) = (0, 0, –1) $\Rightarrow a^2 + b^2 + c^2 = 1$

(a,b,c) = (2,2,3) $\Rightarrow (a^2 + b^2 + c^2) = 17$

so sum of all possible values of $a^2 + b^2 + c^2$ is 18.

7. A point P in the interior of a regular hexagon is at distances 8,8,16 units from three consecutive vertices of the hexagon, respectively. If r is radius of the circumscribed circle of the hexagon,what is the integer closest to r?

7. **(14)**

$\angle DOC = 90^\circ$, $OD = \sqrt{r^2 - \left(\frac{r}{2}\right)^2} = \frac{r\sqrt{3}}{2}$

In rt. $\triangle OPC$, $\quad OP = \sqrt{16^2 - r^2}$ ... (1)

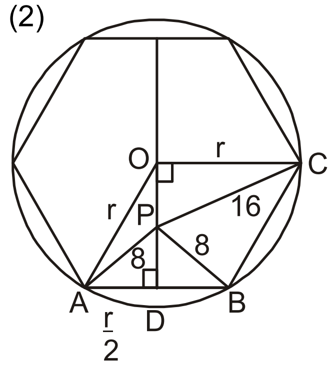
In rt. $\triangle APD$, $\quad PD = \sqrt{8^2 - \left(\frac{r}{2}\right)^2}$

$\therefore \quad OP = OD - PD = \frac{r\sqrt{3}}{2} - \sqrt{64 - \frac{r^2}{4}}$ ... (2)

From (1) & (2)

$$\sqrt{256 - r^2} = \frac{r\sqrt{3}}{2} - \frac{\sqrt{256 - r^2}}{2}$$

$$\Rightarrow \quad \frac{3}{2}\sqrt{256 - r^2} = \frac{r\sqrt{3}}{2}$$



$\Rightarrow \quad 3\left(256-r^2\right)=r^2$

$\Rightarrow \quad r^2=\frac{3\times 256}{4}=r=8\sqrt{3}$

$\therefore$ Integer closet to r is 14.

8. Let AB be a chord of a circle with centre O. Let C be a point on the circle such that $\angle ABC = 30^\circ$ and O lies inside the triagle ABC. Let D be a point on AB such that $\angle DCO = \angle OCB = 20^\circ$. Find the measure of $\angle CDO$ in degreees.

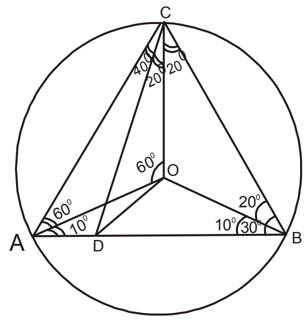
8. **(80)**

Join O to A, $\angle COA = 60^\circ$

ΔCOA is equilateral

$\therefore OC = AC = OA$

$\angle OAB = \angle OBA = 10^\circ$ and $\angle ADC = 70^\circ$



$\therefore AC = CD \qquad \therefore CD = OC$

$\therefore \angle CDO = \angle COD = 80^\circ$

9. Suppose a,b are integers and a+ b is a root of $x^2 + ax + b = 0$. What is the maximum possible value of b^2?

9. (81)

$(a+b)^2 + a(a+b) + b = 0$

$2a^2 + 3ab + b^2 + b = 0$

Since $a \in I$, D must be perfect square

$9b^2 - 8\left(b^2+b\right) = k^2, k \in I$

$b^2 - 8b = k^2 \Rightarrow (b-4)^2 - k^2 = 16$

$\Rightarrow (b-4+k)(b-4-k) = 16$

Let b – 4 + k = α, b – 4 – k = β, then $b=\frac{\alpha+\beta}{2}+4, \alpha\beta = 16$

$\therefore$ maximum value of b^2

$=(\frac{2+8}{2}+4)^2 = 81$

10. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$.

10. **(24)**

In ΔBGC

$4(x^2 + y^2) = 2[10^2 + m^2]$

$2(x^2 + y^2) = 100 + m^2$ ....(1)

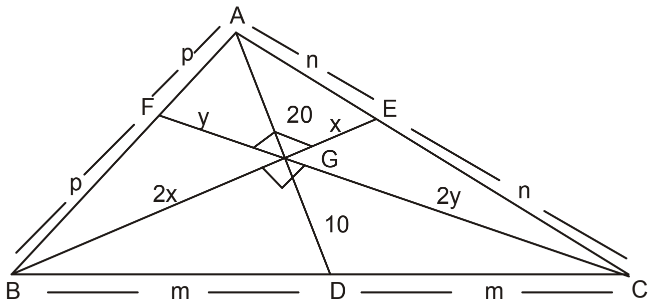
Also, $4(x^2 + y^2) = 4m^2$

$\therefore x^2 + y^2 = m^2$ .....(2)

$\therefore$ Putting in equation (1)

$2m^2 = 100 + m^2 \qquad \therefore m^2 = 100$

In ΔAGC



$(20)^2 + (2y)^2 = 2\left[x^2 + n^2\right] \qquad ...(2)$

In ΔBGA

$(20)^2 + (2x)^2 = 2\left[y^2 + p^2\right] \qquad ...(3)$

Adding (2) and (3)

$800 + 4[x^2 + y^2] = 2[x^2 + y^2 + n^2 + p^2]$

$800 + 2(x^2 + y^2) = 2(n^2 + p^2)$

$\therefore 800 + 2 \times 100 = 2(n^2 + p^2)$

$\therefore n^2 + p^2 = 500$

$$\frac{AB^2 + BC^2 + CA^2}{100} = \frac{4(m^2 + n^2 + p^2)}{100} = \frac{4(600)}{100} = 24$$

11. There are several tea cups in the kitchen, some with handles and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?

11. (29)

Let 'x' the no. of cups without handle & 'y' be the no. of cups with handle.

$$\frac{x(x-1)}{2} \cdot \frac{y(y-1)(y-2)}{3!} = 1200$$

$$\therefore \qquad x(x-1)y(y-1)(y-2) = 14400 = 5!\,5!$$

For maxinum value of x + y; x = 25, y = 4.

12. Determine the number of 8-tuples $(\in_1, \in_2, ..., \in_8)$ such that $\in_1, \in_2, ..., \in_8 \in \{1, -1\}$ and $\in_1 + 2\in_2 + 3\in_3 + ... + 8\in_8$ is a multiple of 3.

12. (88)

$3 | \in_1 + 2\in_2 + ... + 8\in_8$

$3 | \in_1 - \in_2 + \in_4 - \in_5 + \in_7 - \in_8$

Let S = $\in_1 - \in_2 + \in_4 - \in_5 + \in_7 - \in_8$

then S = 0 or ± 6

Each of S = 6 and S = –6 has only one solution.

S = 0 $\Rightarrow$ Of the 6 terms in S, 3 should be 1 and three should be –1.

Which has ${}^6C_3 = 20$ solutions.

So total number of solutions = $(20 + 2) \times 2 \times 2 = 88$ $\quad$ ($\because \in_3$ & $\in_6$ can take any value)

13. In a triangle ABC, right angled at A, the altitude through A and the internal bisector of $\angle A$ have lengths 3 and 4, respectively. Find the length of the median through A.

13. (24)

In ΔADC, $\sin(45^\circ - \theta) = \dfrac{3}{y} \qquad ....(1)$

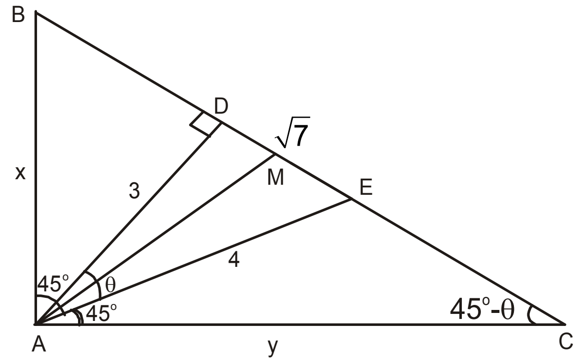
In ΔADE, $\sin\theta = \dfrac{\sqrt{7}}{4}, \cos\theta = \dfrac{3}{4}$

$\therefore$ From equation (1)

$$\frac{1}{\sqrt{2}}[\cos\theta - \sin\theta] = \frac{3}{y}$$

$$\frac{1}{\sqrt{2}}\left[\frac{3}{4} - \frac{\sqrt{7}}{4}\right] = \frac{3}{y} \qquad \therefore y = \frac{12\sqrt{2}}{3-\sqrt{7}}$$

In ΔABD, $\cos(45^\circ - \theta) = \frac{3}{x}$



$$\frac{1}{\sqrt{2}}\left[\frac{3}{4} + \frac{\sqrt{7}}{4}\right] = \frac{3}{x}$$

$$x = \frac{12\sqrt{2}}{3+\sqrt{7}}$$

Let AM be median,

$$x^2 + y^2 = 2[AM^2 + (CM)^2]$$

$$\therefore 2AM^2 = \frac{x^2+y^2}{2} \qquad (\because CM = AM)$$

$$AM = \frac{1}{2}\sqrt{x^2+y^2}$$

$$= \frac{1}{2}\sqrt{\left(\frac{12\sqrt{2}}{3-\sqrt{7}}\right)^2 + \left(\frac{12\sqrt{2}}{3+\sqrt{7}}\right)^2}$$

$$= \frac{1}{2}\sqrt{288\left[\frac{32}{4}\right]} = 24$$

14. If $x = \cos 1^\circ \cos 2^\circ \cos 3^\circ ... \cos 89^\circ$ and $y = \cos 2^\circ \cos 6^\circ \cos 10^\circ ... \cos 86^\circ$. then what is the integer nearest to $\frac{2}{7}\log_2(y/x)$?

14. (19)

$x = \cos 1^\circ \cos 2^\circ ... \cos 89^\circ$

$$x = (\cos 1^\circ \sin 1^\circ)(\cos 2^\circ \sin 2)....(\cos 44^\circ \sin 44^\circ)\cos 45^\circ$$

$$x = \frac{1}{2^{44}}(\sin 2^\circ \sin 4^\circ ... \sin 88^\circ)\cos 45^\circ$$

$$= \frac{1}{2^{44}}[(\sin 2^\circ \sin 88^\circ)(\sin 4^\circ \sin 86^\circ)...(\sin 44^\circ)(\sin 46^\circ)]\cos 45^\circ$$

$$= \frac{1}{2^{44}}[\sin 2^\circ \cos 2^\circ(\sin 4^\circ \cos 4^\circ)...\sin 44^\circ \cos 44^\circ]\cos 45^\circ$$

$$= \frac{1}{2^{44}} \cdot \frac{1}{2^{22}}[\sin 4^\circ \sin 8^\circ ... \sin 88^\circ]\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2^{66} \cdot \sqrt{2}}[\cos 2^\circ \cos 6^\circ ... \cos 86^\circ]$$

$$\frac{y}{x}=2^{66+\frac{1}{2}}=2^{\frac{133}{2}}$$

$$\therefore \frac{2}{7}\times\frac{133}{2}=19$$

15. Let a and b be natural numbers such that 2a – b,a – 2b and a + b are all distinct squares. What is the smallest possible value of b?

15. **(21)**

Let 2 a – b = m^2, a – 2b = n^2 and a + b = p^2, where $m,n,p\in I$ then

$m^2 - n^2 = a + b = p^2$ ....(1)

Also, $a=\frac{m^2+p^2}{3}$ and $b=\frac{p^2-n^2}{3}$

$\therefore 3\,|\,m^2+p^2$ and $3\,|\,p^2-n^2$ ....(2)

so b is smallest when (m,n,p) = (15, 9, 12).

$\therefore$ smallest value of b = $\frac{12^2-9^2}{5}=21$

16. What is the value of $\sum\limits_{\substack{1\le i<j\le 10\\ i+j=odd}}(i+j)-\sum\limits_{\substack{1\le i<j\le 10\\ i+j=even}}(i+j)$?

16. **(55)**

$$\sum_{\substack{1\le i<j\le 10\\ i+j=odd}}(i+j)-\sum_{\substack{1\le i<j\le 10\\ i+j=even}}(i+j)$$

$$=(3+2.5+3.7+4.9+5.11+4.13+3.15+2.17+19)-$$

$$(4+2.6+3.8+4.10+4.12+3.14+2.16+18)$$

= 275 – 220

= 55.

17. Triangles ABC and DEF are such that $\angle A=\angle D, AB=DE=17, BC=EF=10$ and AC – DF = 12. What is AC + DF ?

17. **(30)**

Let AC = x and DF = y, then AC – DF = x – y = 12. ....(1)

Now applying cosine rule,

$$\cos(\angle A)=\cos(\angle D)$$

$$\Rightarrow \frac{17^2+x^2-10^2}{2\times17\times x}=\frac{17^2+y^2-10^2}{2\times17\times y}$$

$$\Rightarrow \frac{189+x^2}{x}=\frac{189+y^2}{y}=\frac{189+x^2-(189+y^2)}{x-y}=x+y$$

$$\Rightarrow 189+x^2=x(x+x-12)\Rightarrow x^2-12x-189=0$$

$$\Rightarrow x=21 \qquad \therefore y=x-12=9$$

$$\therefore AC+DF=x+y=30$$

18. If a, b, c $\geq$ 4 are integers, not all equal, and $4abc=(a+3)(b+3)(c+3)$, then what is the value of a + b + c ?

18. **(16)**

Let $a \le b \le c$, then

$$4 = \frac{(a+3)(b+3)(c+3)}{abc} = \left(1+\frac{3}{a}\right)\left(1+\frac{3}{b}\right)\left(1+\frac{3}{c}\right) \le \left(1+\frac{3}{a}\right)^3$$

$\Rightarrow a \le 5$ $\quad\therefore$ a = 4 or 5

Case I : a = 4,

$$4 = \frac{7}{4}\left(1+\frac{3}{b}\right)\left(1+\frac{3}{c}\right) \le \frac{7}{4}\left(1+\frac{3}{b}\right)^2 \Rightarrow \left(1+\frac{3}{b}\right)^2 \ge \frac{16}{7} \Rightarrow b \le 5 \qquad \therefore b = 4 \text{ or } 5$$

If b = 4, then $c = \frac{49}{5} \notin I$

If b = 5, then c = 7

$\therefore$ (a,b,c) = (4,5,7) is a solution

Case II : a = 5, then

$$4 = \frac{8}{5}\left(1+\frac{3}{b}\right)\left(1+\frac{3}{c}\right) \le \frac{8}{5}\left(1+\frac{3}{b}\right)^2 \Rightarrow \left(1+\frac{3}{b}\right)^2 \ge \frac{5}{2} \Rightarrow b \le 5 \qquad \therefore c = \frac{16}{3} \notin I$$

$\therefore$ a + b + c = 4 + 5 + 7 = 16

19. Let N = 6 + 66 + 666 + ... + 666...66, where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N ?

19. (33)

N = 6 + 66 + 666 + ... to 100 terms

$$= \frac{6}{9}[9 + 99 + 999 + \dots \text{ to } 100 \text{ terms}]$$

$$= \frac{6}{9}[10-1) + (100-1) + (1000-1) + \dots \text{ to } 100 \text{ terms}]$$

$$= \frac{6}{9}[10 + 100 + 1000 + \dots \text{ to } 100 \text{ terms} - 100]$$

$$= \frac{6}{9}\left[\frac{10(10^{100}-1)}{10-1} - 100\right]$$

$$= \frac{6}{9}\left[\underbrace{111\dots1}_{98 \text{ times 1's}}010\right]$$

$$= \frac{1}{3} \times 222\dots2020 = \underbrace{740740\dots7}_{33 \text{ times 7's}}\,374$$

$\therefore$ 7 occurs 33 times in N.

20. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$.

20. (17)

Let n has k digits, then product of digits of $n \le 9^k < 10^k \le 10n$

$\therefore \quad n^2 - 15n - 27 \le 10n$

$\Rightarrow n^2 - 25n - 27 \le 0$

$\Rightarrow n \le 26$

Also, $n^2 - 15n - 27 \ge 0$

$\Rightarrow n \ge 17$

$\therefore$ Product of digits $\leq 2\times 6 = 12$

$\therefore n^2 - 15n - 27 \leq 12 \Rightarrow n(n-15) \leq 39 \Rightarrow n \leq 17$

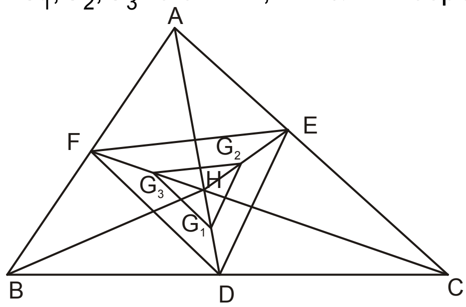
$\therefore n = 17$ Which also satisfies the given condition

$\therefore$ Sum of all possible values of n = 17.

21. Let ABC be an acute-angled triangle and let H be its orthocentre. Let G_1, G_2 and G_3 be the centroids of the triangles HBC, HCA and HAB, respectively. If the area of triangle $G_1G_2G_3$ is 7 units, what is the area of triangle ABC ?

21. **(63)**

Let D,E,Fbe the mid points of BC, CA and AB. Then G_1,G_2,G_3 lie on HD, HE & HF respectively such that $HG_1 : G_1D = HG_2 : G_2E = HG_3 : G_3F = 2 : 1$



$\therefore \Delta HG_1G_2 \sim \Delta HDE$

$$\therefore \frac{ar(\Delta HG_1G_2)}{ar(\Delta HDE)} = \left(\frac{HG_1}{HD}\right)^2 = \frac{4}{9}$$

$$\therefore ar(\Delta HG_1G_2) = \frac{4}{9}\times ar(\Delta HDE)$$

similarly $ar(\Delta HG_2G_3) = \frac{4}{9}\times ar(\Delta HEF)$ and $ar(\Delta HG_3G_1) = \frac{4}{9}\times ar(\Delta HFD)$

$$\therefore ar(\Delta G_1G_2G_3) = \frac{4}{9}\times ar(\Delta DEF) = \frac{4}{9}\times\frac{1}{4}ar(\Delta ABC)$$

$\therefore ar(\Delta ABC) = 9\times ar(\Delta G_1G_2G_3) = 9\times 7 = 63$ units

22. A positive integer k is said to be good if there exists a partition of {1, 2, 3, ... 20} in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there ?

22. **(06)**

$$1 + 2 + 3 + ... + 20 = \frac{20\times 21}{2} = 210 = 2\times 3\times 5\times 7$$

So number of partitions must be a factor of 210. If number of partitions is greater than 10, then at least two partitions must be singleton and hence can not have equal sums. So number of partitions can be 2, 3, 5, 6, 7 and 10 only. Each case can be easily verfied to true. Hence there are 6 corresponding good numbers.

23. What is the largest positive integer n such that

$$\frac{a^2}{\frac{b}{29}+\frac{c}{31}} + \frac{b^2}{\frac{c}{29}+\frac{a}{31}} + \frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq n(a+b+c)$$

holds for all positive real numbers a, b, c.

23. **(14)**

Taking a = b = c,

$$3\times\frac{29\times 31a}{29+31} \geq n\times 3a$$

$$\Rightarrow n \leq \frac{29\times 31}{60} \Rightarrow n \leq 14.$$

$$\text{Let } S = \frac{a^2}{\frac{b}{29}+\frac{c}{31}} + \frac{b^2}{\frac{c}{29}+\frac{a}{31}} + \frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq 28\left[\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}\right]$$

Let x = b + c, y = c + a, z = a + b, then $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=\frac{(y+z-x)^2}{4x}+\frac{(z+x-y)^2}{4y}+\frac{(x+y-z)}{4z}$

$$=\frac{(x+y+z)^2}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-\frac{x(y+z)}{x}-\frac{y(z+x)}{y}-\frac{z(x+y)}{z}$$

$$=\frac{(x+y+z)}{4}(9-8) \qquad \left(\because \frac{x+y+z}{3}\geq\frac{3}{\frac{1}{x}+\frac{1}{9}+\frac{1}{2}}\right)$$

$$=\frac{1}{2}(a+b+c)$$

$\therefore S\geq 14(a+b+c)$, Hence largest value of n is 14.

24. If N is the number of triangles of different shapes (i.e., not similar) whose angles are all integers (in degrees), what is N/100?

24. (27)

Let the angles be A,B,C in degrees.

Case I : A = B = C = 60

Case II : $A=B\neq C$

Then 2A + C = 180 $\Rightarrow$ C = 180 – 2A

$\therefore$ $1\leq A\leq 89$ but $A\neq 60$

$\therefore$ 88 such triangles are possible.

Case III : $1\leq A<B<C$

A + B + C = 180

$\therefore$ No. of triangles in this case

$$=\frac{{}^{180-3+2}C_2-88\times 3-1}{\lfloor 3}=2611$$

$\therefore$ Total no. of triangles = 2611 + 88 + 1 = 2700 = N.

$\therefore$ $\frac{N}{100}=27.$

25. Let T be the smallest positive integer which, when divided by 11, 13, 15 leaves remainders in the sets {7, 8, 9}, {1, 2, 3}, {4, 5, 6} respectively. What is the sum of the squares of the digits of T ?

25. (81)

T + 26 should give remainders in the sets {0,1,2}, {1,2,3} and {0,1,2} when divided by 11, 13 and 15 respectively.

$\because 15k+r\equiv(4k+r)$ mod11 and $15k+r\equiv(2k+r)$ mod 13. Checking remainders of 4k, 4k + 1, 4k + 2, $k\in I$ when divided by 11, we get k can be 3,5,6,8,11,14,... Now checking remainders of 2k, 2k + 1, 2k + 2 when divided by 13, k can be 6,14,... On checking we find least value of T + 26 (greater than 26) is obtained for k = 14, r = 0.

$\therefore T+26=210\Rightarrow T=184$

$1^2+8^2+4^2=81$

26. What is the number of ways in which one can choose 60 unit squares from a 11 × 11 chessboard such that no two chosen squares have a side in common ?

26. (62)

Maximum number of squares no two of which are adjacent can be obtained only when we choose the squares alternately along each row and each column. Starting from a corner square we will get a maximum of 61 squares of which no two will be adjacent. Now if we remove any 1 square from the 61 squares, we are left with 60 square with the required properly. Again if we start from a square next to a corner square and choose alternately, we agian get 60 squares with required property. So in total 62 ways are there to choose the 60 squares.

27. What is the number of ways in which one can colour the squares of a 4 × 4 chessboard with colours red and blue such that each row as well as each column has exactly two red squares and two blue squares ?

27. **(90)**

First row can be coloured in $^4C_2$ ways. The column of first red square can be filled in $^3C_1$ ways . So in total $^4C_2 \times {}^3C_1 = 18$ ways

After which there are two possibilities

Case - I In this case remaning squares can be filled in only one way

R	R		
R	R		

Case - II

In this case remaining squares can be filled in 4 ways

R	R		
R		R	

R	R		
R			R

So total no. of ways = 18 (1 + 4) = 90

28. Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N.

28. **(24)**

Case - I

Children are given 1,2 and 5 chocolates. No. of ways $= \frac{8!}{1!2!5!} \times 3! = 1008$

Case- II

Children are given 1,3 and 4 chocolates. No. of ways $= \frac{8!}{1!3!4!} \times 3! = 1680$

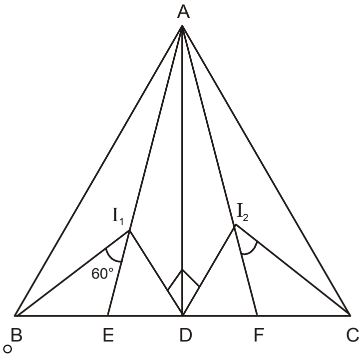
$\therefore$ Total no. of ways = 2688 = N

$\therefore$ Sum of digits of N = 24

29. Let D be an interior point of the side BC of a triangle ABC. Let I_1 and I_2 be the incentres of triangles ABD and ACD respectively. Let AI_1 and AI_2 meet BC in E and F respectively. If $\angle BI_1E = 60°$, what is the measure of $\angle CI_2F$ in degrees ?

29. **(30)**

$\angle BI_1A = 180° - 60° = 120°$



$\therefore \quad = 90° + \frac{1}{2}\angle ADB$

$\therefore \quad \angle ADB = 60°$

$\therefore \quad \angle ADC = 180° - 60° = 120°$

$\therefore \quad \angle AI_2C = 90° + \frac{1}{2}\angle ADC = 150°$

$\therefore \quad \angle CI_2F = 180° - \angle AI_2C = 180° - 150° = 30°$

30. Let $P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ be a polynomial in which a_i is a non-negative integer for each $i \in \{0, 2, 3, ..., n\}$. If P(1) = 4 and P(5) = 136, what is the value of P(3)?

30. (34)

$P(1) = a_0 + a_1 + a_2 + ... + a_n = 4$ ... (1)

and $P(5) = a_0 + 5a_1 + 25a_2 + 125a_3 + ... + 5^n a_n = 136$ ... (2)

$\because \quad a_i \ge 0 \ \forall i$

$\therefore$ from (2), $a_i = 0 \ \forall i > 3$

$\therefore \quad a_0 + a_1 + a_2 + a_3 = 4$

$\Rightarrow 0 \le a_0, a_1, a_2, a_3 \le 4$

Also, $a_0 + 5a_1 + 25a_2 \le 124$

So we must have $a_3 = 1$.

which gives $a_0 + 5a_1 + 25a_2 = 11$.

So $a_2 = 0$

$\therefore \quad a_0 + a_1 = 3$ and $a_0 + 5a_1 = 11$

which gives $a_0 = 1, a_1 = 2$

$\therefore \quad P(x) = 1 + 2x + x^3$

$\therefore \quad P(3) = 34$.

ANSWER KEY (PRMO-2018)

1. (17)	2. (08)	3. (70)	4. (12)	5. (84)
6. (18)	7. (14)	8. (80)	9. (81)	10. (24)
11. (29)	12. (88)	13. (24)	14. (19)	15. (21)
16. (55)	17. (30)	18. (16)	19. (33)	20. (17)
21. (63)	22. (06)	23. (14)	24. (27)	25. (81)
26. (62)	27. (90)	28. (24)	29. (30)	30. (34)