33rd Indian National Mathematical Olympiad - 2018

Time: 4 hours **January 21,2017**

Instructions :

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer all the questions.s**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let ABC be a non-equilateral triangle with integer sides. Let D and E be respectively the mid-points BC and CA; let G be the centroid of triangle ABC. Suppose D,C,E,G are concyclic. Find the least perimeter of triangle ABC.

2. For any natural number n, consider a $1\times n$ rectangular board made up of n unit squares. This is covered by three types of tiles: 1×1 red tile, 1×1 green tile and 1×2 blue domino. (For example, we can have 5 types of tiling when n = 2: red-red; red-green; green-red; green-green; and blue.) Let t_n denote the number of ways of covering $1\times n$ rectangular board by these types of tiles. Prove that t_n divides t_{2n+1}.

3. Let Γ_1 and Γ_2 be two circles with respective centres O_1 and O_2 intersecting in two distinct points A and B such that $\angle O_1AO_2$ is an obtuse angle. Let the circumcircle of triangle O_1AO_2 intersect Γ_1 and Γ_2 respectively in points $C(\neq A)$ and $D(\neq A)$. Let the line CB intersect Γ_2 in E; let the line DB intersect Γ_1 in F. Prove that the points C,D,E,F are concyclic.

4. Find all polynomials with real coefficients P(x) such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

5. There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbors combined, the teacher takes away one apple from that girl and gives one apple each to her neighbors. Prove that this process stops after a finite number of steps. (Assume that the teacher has an abundant supply of apples.)

6. Let N denote the set of all natural numbers and let $f : N \to N$ be a function such that

(a) $f(mn) = f(m)f(n)$ for all m,n in N;

(b) m + n divides f(m) + f(n) for all m,n in N.

Prove that there exists an odd natural number k such that $f(n) = n^k$ for all n in N.

SOLUTION

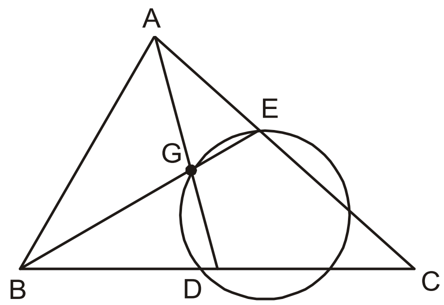
1. Let ABC be a non-equilateral triangle with integer sides. Let D and E be respectively the mid-points BC and CA; let G be the centroid of triangle ABC. Suppose D,C,E,G are concyclic. Find the least perimeter of triangle ABC.

Sol. $\because$ D, C,E,G are concyclic

$\therefore \quad AG \cdot AD = AE \cdot AC$

But $AG = \frac{2}{3}AD$ $\quad$ ($\because$ G is centroid)

and $AE = \frac{AC}{2}$ $\quad$ ($\because$ E is mid point)



$\therefore \quad \frac{2}{3}AD \cdot AD = \frac{AC}{2} \cdot AC$

$\Rightarrow AD^2 = \frac{3}{4}AC^2$

Now, AD is median

$\therefore \quad AB^2 + AC^2 = 2(AD^2 + BD^2)$

$\Rightarrow AB^2 + AC^2 = 2\left(\frac{3}{4}AC^2 + \frac{BC^2}{4}\right) \Rightarrow 2AB^2 = BC^2 + AC^2$

i.e $a^2 + b^2 = 2c^2$ $\quad$ ....(i)

where a = BC, b = AC and C = AB are positive integers.Without loss of generality, we may assume a > b.Since $a^2 + b^2$ is even, therefore a and be must have same pairing.

$\therefore$ from (i), $\left(\frac{a-b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 = c^2$

Hence, $\left(\frac{a-b}{2}, \frac{a+b}{2}, c\right)$ from a Pythagorean triple.

$\therefore \frac{a-b}{2} = k\left(m^2 - n^2\right), \frac{a+b}{2} = 2kmn$ and $c = k(m^2 + n^2)$ for some positive integers k,m,n with $m > n$.

$\therefore \quad a + b + c = k(m^2 + n^2 + 4mn)$

but $a - b < c \Rightarrow 2k(m^2 - n^2) < k(m^2 + n^2) \Rightarrow m^2 < 3n^2 \Rightarrow n \geq 2, m \geq 3$ $(\because m > n)$

So the least possible perimeter is 37 for k = 1, m = 3, n = 2

2. For any natural number n, consider a $1\times n$ rectangular board made up of n unit squares. This is covered by three types of tiles: 1×1 red tile, 1×1 green tile and 1×2 blue domino. (For example, we can have 5 types of tiling when n = 2: red-red; red-green; green-red; green-green; and blue.) Let t_n denote the number of ways of covering $1\times n$ rectangular board by these types of tiles. Prove that t_n divides t_{2n+1}.

Sol. Clearly $t_1 = 2$ & $t_2 = 5$. For each $n \ge 3,$ if the last unit square is covered with red or green tile, then the remaining n -1 squares can be covered in t_{n-1} ways. Again, if the last two unit squares are covered with a blue domino, then the remaining n – 2 squares can be covered in t_{n-2} ways.

$\therefore t_n = 2t_{n-1} + t_{n-2}\ \forall n \in \{3,4,5,...\}.$ ....(1)

Let for each $k \in \{1,2,...,n-1\}$, $s_k = t_{n+k} + (-1)^n t_{n-k}$, then

$s_1 = t_{n+1} - t_{n-1} = (2t_2 + t_{n-1}) - t_{n-1} = 2t_n$ ....(2)

$s_2 = t_{n+2} + t_{n-2} = (2t_{n+1} + t_n) + (t_n - 2t_{n-1})$

$= 2(t_{n+1} - t_{n-1}) + 2t_n$

$= 4t_n + 2t_n$ (using(2))

$= 6t_n$ ...(3)

If k > 1is odd , then

$s_k = t_{n+k} - t_{n-k} = (2t_{n+k-1} + t_{n+k-2}) - (t_{n-k+2} - 2t_{n-k+1})$

$= 2(t_{n+k-1} + t_{n-k+1}) + (t_{n+k-2} - t_{n-k+2})$

$= 2s_{k-1} + s_{k-2}$

If k > 2 is even, then

$s_k = t_{n+k} + t_{n-k} = (2t_{n+k-1} + t_{n+k-2}) + (t_{n-k+2} - 2t_{n-k+1})$

$= 2(t_{n+k-1} - t_{n-k+1}) + (t_{n+k-2} + t_{n-k+2}) = 2s_{k-1} + s_{k-2}$

Thus, $s_k = 2s_{k-1} + s_{k-2} \forall\ k \in \{3,4,5,...,n-1\}.$ ...(4)

Now from (2) and (3), $t_n \mid s_1$ and $t_n \mid s_2$ and from (4), $t_n | s_k$ whenever $t_n | s_{k-1}$ and $t_n | s_{k-2}$.

So by induction, $t_n \mid s_k \forall k \in \{1,2,...,n-1\}$

Now $t_{2n+1} = 2t_{2n} + t_{2n-1}$

$= 2(2t_{2n-1} + t_{2n-2}) + t_{2n-1}$

$= 5t_{2n-1} + 2t_{2n-2}$

$= 5(t_{2n-1} + (-1)^{n-1} t_1) + 2(t_{2n-2} + (-1)^{n-2} t_2) - (-1)^{n-1}(5t_1 - 2t_2)$

$= 5s_{2n-1} + 2s_{2n-2}$ $(\because t_1 = 2, t_2 = 5)$

$\therefore t_n \mid t_{2n+1}$ Q.E.D.

3. Let Γ_1 and Γ_2 be two circles with respective centres O_1 and O_2 intersecting in two distinct points A and B such that $\angle O_1AO_2$ is an obtuse angle. Let the circumcircle of triangle O_1AO_2 intersect Γ_1 and Γ_2 respectively in points $C(\neq A)$ and $D(\neq A)$. Let the line CB intersect Γ_2 in E; let the line DB intersect Γ_1 in F. Prove that the points C,D,E,F are concyclic.

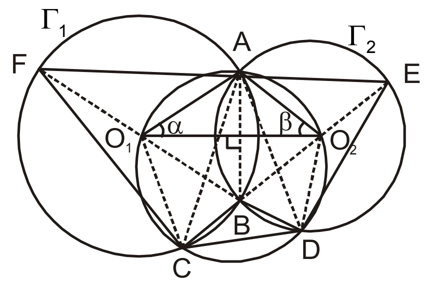
Sol. Join O_1B, O_2B, O_1C, O_2D, AC, AD and AB.

Let $\angle AO_1O_2 = \alpha$ and $\angle AO_2O_1 = \beta$.

$\because$ AB is common chord of circles Γ_1 and Γ_2

$\therefore O_1O_2 \perp AB$

Now in circumcircle of ΔO_1AO_2,



$\angle O_1CA = \angle O_1O_2A = \beta$

$\angle O_1CA = \angle O_1AO_2 = \beta$ $(\because O_1A = O_1C)$

$\therefore \angle BAC = \frac{\pi}{2} - (\angle AO_1O_2 + \angle O_1AC) = \frac{\pi}{2} - (\alpha + \beta)$

$\therefore \angle BO_1C = 2\angle BAC$ (using circle Γ_1)

$= \pi - 2(\alpha + \beta)$ .....(i)

Similarly, we can show that

$\angle BAD = \frac{\pi}{2} - (\alpha + \beta)$

$\therefore \angle BO_2D = 2\angle BAD = \pi - 2(\alpha + \beta)$ ....(ii)

Now $\angle CAD = \angle CAB + \angle DAB = \pi - 2(\alpha + \beta)$

$\therefore \angle CO_1D = \angle CO_2D = \angle CAD = \pi - 2(\alpha + \beta)$ ....(iii)

From (i), (ii) and (iii)

$\therefore \angle CO_1B = \angle CO_1D$ and $\angle DO_2B = \angle DO_2C$

$\therefore D, B, O_1$ are collinear and C, B, O_2 are collinear.

$\therefore$ O_1 and O_2 lie on DF and DE respectively.

So $\angle DFC = \angle BFC = \frac{1}{2}\angle BO_1C = \frac{1}{2}\angle BO_2D = \angle DEB = \angle DEC$

Hence, C,D,E,F are concyclic.

4. Find all polynomials with real coefficients P(x) such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

Sol. $\because$ $P(x^2 + x + 1)$ divides $P(x^3 - 1)$

$\therefore$ $P(x^3 - 1) = P(x^2 + x + 1)\, Q(x)$ ....(i)

where Q(x) is some non-zero polynomial. If $P(x) \equiv c$, a constant, then equation (i) is clearly satisfied for Q(x) = 1. Now we assume that P(x) is not a constant function. Let $w^2 + w + 1$ be a complex root of P(x) having maximum possible modulus. Put x = w in equation (i), so that

$$P(w^3 - 1) = P(w^2 + w + 1)Q(w) = 0$$

$\therefore w^3 - 1$ is also a root of P(x). Let $w_1 = w^3 - 1$.

Again put x = –1 – w in equation (1), so that

$$P((-1 - w^3) - 1) = P(w^2 + w + 1)Q(w) = 0$$

$\therefore (-1 - w)^3 - 1$ is also a root of P(x). Let $w_2 = (-1 - w)^3 - 1$.

Now, $|w_1 + w_2| = |w^3 - 1 + (-1 - w)^3 - 1| = 3|w^2 + w + 1|$ ....(2)

But $|w_1 + w_2| \le |w_1| + |w_2| \le |w^2 + w + 1| + |w^2 + w + 1| = 2|w^2 + w + 1| \le 3|w^2 + w + 1|$

So equalitity (2) is possible only if

$$w_1 = w_2 = w^2 + w + 1 = 0$$

But $w^2 + w + 1$ is the root of maximum modulus,

hence all roots of P(x) must be equal to zero.

$\therefore$ $P(x) = cx^n$, where $c \in R - \{0\}$ and $n \in N$.

Which also satisfies equation (1).

Thus $P(x) \equiv c$ and $P(x) = cx^n$ are the only polynomials satisfying the given condition.

5. There are $n \ge 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbors combined, the teacher takes away one apple from that girl and gives one apple each to her neighbors. Prove that this process stops after a finite number of steps. (Assume that the teacher has an abundant supply of apples.)

Sol. Let us suppose that the process does not stope in finite number of steps. Let $G_1, G_2, ..., G_n$ be the n girls having $m_{1k}, m_{2k}, ..., m_{nk}$ apples at the end of k^{th} step, $k \in \{0, 1, 2, ...\}$, where k = 0 denotes initial situation. Let $M_k = \max\{m_{1k}, m_{2k}, ..., m_{nk}\}$ and $S_k = m_{1k} + m_{2k} + ... + m_{nk}$ for each $k \in \{0, 1, 2, ...\}$. Then $S_k \le M_k + M_k + ...\text{n times} = nM_k \ \forall k \in \{0, 1, 2, ...\}$.

Let in the k^{th} step some girl G_i has more apples than the combined total of her neighbors, then both her neighbors must have less apples than her. Now if one apple is taken from G_i and one apple is given to each of her neighbors, then each of the neighbors can have at most $m_{i,k-1}$ apples and $m_{ik} = m_{i,k-1} - 1$. Thus the maximum number of apples M_{k-1} either remaine the same or decreases by 1.

$\therefore M_k \le M_{k-1} \ \forall k \in \{1, 2, 3, ...\}$.

$\therefore$ by induction, $M_k \le M_{k-1} \le M_{k-2} \le .... \le M_o$

$\therefore$ $S_k \le nM_k \le nM_o \forall k \in \{0, 1, 2, ...\}$. ....(1)

But after each step one apple is taken from a girl and two apples are given to neighbors.

So $S_k = S_{k-1} + 1$.

Thus $S_0, S_1, S_2, ...$ is an strictly increasing sequence of positive integers and hence is bounded. Which contradicts (1), so the process must end in a finite number of steps.

6. Let N denote the set of all natural numbers and let $f: N \to N$ be a function such that

(a) $f(mn) = f(m)f(n)$ for all m,n in N;

(b) m + n divides f(m) + f(n) for all m,n in N.

Prove that there exists an odd natural number k such that $f(n) = n^k$ for all n in N.

Sol. Given f(mn) =f(m) f(n) $\forall m,n \in N$ ...(1)

and m + n| f(m) + f(n) $\forall m,n \in N$ ...(2)

Putting m = n = 1 in (1),

$f(1) = f(1)^2 \Rightarrow f(1) = 1$

Putting m = n = 2 in (2),

$4 \mid 2f(2) \Rightarrow 2 \mid f(2)$

Let $f(2) = 2^k p$, where p is odd.

Now from (2),

$(p-1) + 1 \mid f(p-1) + f(1) \Rightarrow p \mid f(p-1) + 1$ ...(3)

Again, $f(p-1) = f\left(2 \cdot \frac{p-1}{2}\right) = f(2).f\left(\frac{p-1}{2}\right) = 2^k p\, f\left(\frac{p-1}{2}\right)$

$\therefore p \mid f(p-1)$ ...(4)

From (3) and (4),

$p \mid f(p-1) + 1 - f(p-1) \quad \Rightarrow p \mid 1 \Rightarrow p = 1$

$\therefore f(2) = 2^k$.

Again putting m = 2, n = 1 in (2),

$2+1 \mid f(2) + f(1) \Rightarrow 3 \mid 2^k + 1 \Rightarrow 3 \mid (3-1)^k + 1 \Rightarrow$ k is odd.

From (1),

$f(n^m) = f(n^{m-1} \cdot n) = f(n^{m-1}) \cdot f(n)$

$\therefore$ by induction $f(n^m) = (f(n))^m$

Now, from (2),

$2^m + n \mid f(2^m) + f(n)$

$\Rightarrow 2^m + n \mid f(2)^m + f(n) \Rightarrow 2^m + n \mid 2^{km} + f(n)$

$\Rightarrow 2^m + n \mid ((2^m)^k + n^k) + (f(n) - n^k) \quad \Rightarrow 2^m + n \mid f(n) - n^k$

But m is arbitrary. This means $f(n) - n^k$ has infinite divisor $2^{m+n}, m \in \{1, 2, ...\}$, which is possible only if $f(n) - n^k = 0$

$\therefore f(n) = n^k$.